A Simple Formula of Current in Dipole Antennas

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Abstract—**A simple and quantitatively accurate representation of the current distribution in a dipole antenna is derived. Numerical data are given and are found to be in good agreement with the experiment when $h \geqq 0.15\lambda$.**

I. Introduction

The PROBLEM of the thin dipole antenna, whether center-driven or asymmetrically driven, for antenna lengths that are short, long, or infinitely long, has already been treated by many authors [1]–[15]. For example, King–Middleton's iteration procedure [4] gives accurate input admittances; the so-called three-term theory [12] provides simple expressions of current distributions on antennas less than a wavelength long; the Wiener–Hopf procedure [10] predicts successfully the behavior of longer dipole antennas; and, if more accurate theoretical data are desired, solving the problem numerically on a high-speed computer is now quite feasible [11], [13].

The purpose of this paper is to give, for practical reasons, a theory of the dipole antenna which can give a simple yet accurate algebraic expression for the current distribution everywhere on the antenna even when antennas are not short or are asymmetrically driven. Such an expression is found, provided the generator is not less than 0.15 wavelength from either end and the antenna is not too thick. Since the formula is simple and general, it is particularly useful in discussing the properties of antennas with multiple loadings or excitations or whenever a superposition of current distributions is needed, such as in the case of calculating the transient response of a dipole to a short pulse applied at its driving point. These possibilities are being studied. Note that in the work of King and Sandler [14], trigonometric functions are used to represent currents on long antennas. However, their theory is specifically for resonant dipoles that are not excessively long. There are no such limitations in the present theory. Although the present theory follows the approach which was used by Hallen [5], [8], and later by Chen and Keller [15], the results obtained here are quantitatively accurate [16] and new.

II. Current on an Infinitely Long Antenna

The current distribution on an infinitely long antenna is given as follows:

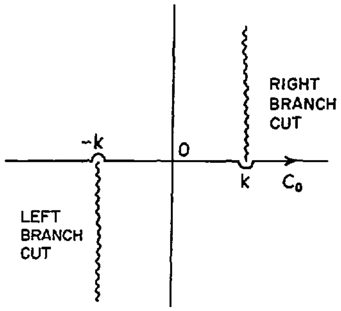

Fig. 1. The branch cuts and contour on the ζ-plane.

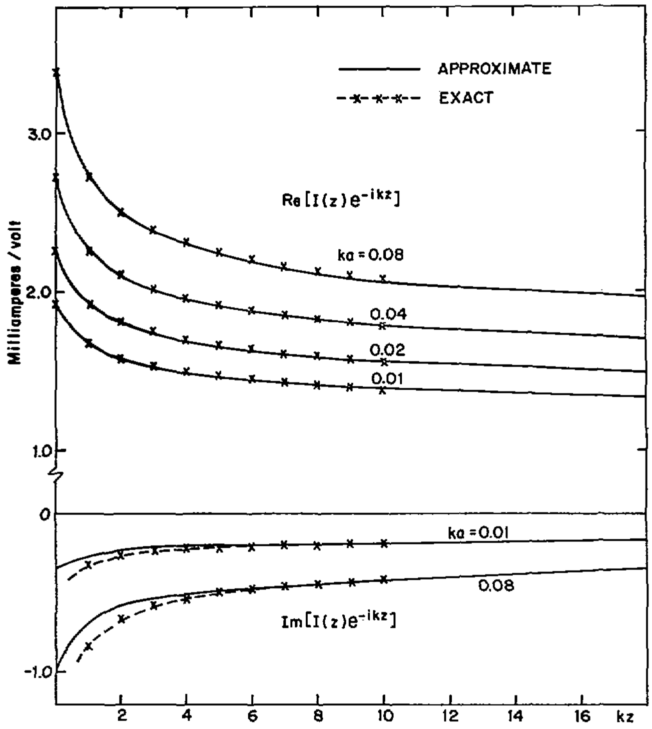

Fig. 2. Current distribution on an infinitely long antenna.

$$I_\infty(z) = \frac{2ik}{\zeta_0} \int_{C_0} \frac{e^{i\zeta z}}{(k^2 - \zeta^2)\overline{K}(\zeta)} d\zeta, \qquad (1)$$

where $\overline{K}(\zeta) = \pi i J_0 H_0^{(1)}$ and J_0 and $H_0^{(1)}$ are Bessel functions with arguments $a\,(k^2-\zeta^2)^{1/2}$. The branch cuts are shown in Fig. 1, together with the contour C_0. ζ_0 is the intrinsic impedance of the free space. Numerical integration of (1) was carried out by Duncan [17], and an asymptotic expression for $I_\infty(z)$ that involves two series was obtained by Kunz [18]. In this section, a simple and accurate expression for $I_\infty(z)$ is derived which contains only one term.

The results of a numerical integration of (1) for different values of kz are shown in Fig. 2. Note that the e^{ikz} factor has been suppressed to reveal the traveling-wave nature of the

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current on the antenna. The phase of the current, when the suppressed e^{ikz} factor is taken into account, is seen to be very close to a pure traveling wave, even when kz is small. In other words, it is seen that the effect of the generator is localized. This observation is very useful to explain the final conclusion of this study, that the current on a finite dipole can be represented by outgoing and reflected waves even when the length of the dipole is not very long.

Since $I_\infty(z)$ plays a very important role in the present theory, it is necessary to find a simple representation for it. For this purpose, the contour C_0 is deformed into C_2 so that it is wrapped around the right branch cut. Suppose that kz is not too small so that the contribution to the integral comes from near $\zeta = k$; $I_\infty(z)$ can be approximated by the following integral:

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_0}\int_0^\infty \frac{e^{-2kz\eta}}{\eta}\left[\frac{1}{2C_w - \log\eta + i\dfrac{3}{2}\pi} - \frac{1}{2C_w - \log\eta - i\dfrac{\pi}{2}}\right]d\eta, \tag{2}$$

where $C_w = \log(1/ka) - \gamma$, and $\gamma = 0.577\,215\,66$. Due to the exponential factor, the limit of the integration can be "cut off" to some number D' which is approximately $e^{-\gamma}$ (see Appendix):

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_0}\int_0^{D'/2kz}\frac{d\eta}{\eta}\left[\frac{1}{2C_w - \log\eta + i\dfrac{3}{2}\pi} - \frac{1}{2C_w - \log\eta - i\dfrac{1}{2}\pi}\right]. \tag{3}$$

The integral of (3) can be carried out to get

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_0}\log\left[1 - \frac{2\pi i}{2C_w + \log 2kz + \gamma + i\dfrac{3}{2}\pi}\right] \tag{4}$$

for large kz (see (6) for the general case).

The input conductance of the infinitely long antenna is known for small ka [10] to be approximately

$$G_\infty = \frac{1}{\zeta_0}\operatorname{Im}\left\{\log\left(1 + \frac{i\pi}{C_w}\right) + \frac{\pi^2}{12}\left[\frac{1}{(C_w - \log 2)^2} - \frac{1}{(C_w - \log 2 + \pi i)^2}\right]\right\}. \tag{5}$$

In order to obtain a formula for $I_\infty(z)$ which is good even for small kz, the term $\log 2kz$ in (4) is changed to $\log(kz + \sqrt{(kz)^2 + D''})$. The constant D'' is determined by requiring that when $kz = 0$, the real part of $I_\infty(0)$, as given by (4), and G_∞, as given by (5), are not to differ by more than $1/(C_w)^3$ in the limit that ka approaches zero. If this is done, D'' is found to be equal to $e^{-2\gamma}$. Thus for all kz, $I_\infty(z)$ can be expressed as

$$I_\infty(z) = \frac{ie^{ikz}}{\zeta_0}\cdot\log\left[1 - \frac{2\pi i}{2C_w + \log\left(kz + \sqrt{(kz)^2 + e^{-2\gamma}}\right) + \gamma + i\dfrac{3}{2}\pi}\right]. \tag{6}$$

The above equation can be thought of as an interpolating formula between (4) and (5). To illustrate the accuracy, numerical data from (6) are plotted in Fig. 2 together with the exact values. It can be seen that (6) remains in excellent agreement with the exact value until kz is less than unity, where a discrepancy in the imaginary part begins to appear; the real part is good to the driving point.

III. REFLECTED CURRENT AT THE END OF A SEMI-INFINITE ANTENNA

Let $i(z)$ be the current on a semi-infinite antenna with one end at $z = 0$ and the other end at $z = \infty$, and excited by a generator of V volts at $z = L$; then as L and V are allowed to become infinity, so that the amplitude of the incident current is unity, $i(z)$ is given as follows [8]:

$$i(z) = \frac{k}{\pi i}\overline{L}_-(-k)\int_{C_0}\frac{e^{i\zeta z}d\zeta}{(k^2 - \zeta^2)\overline{L}_-(\zeta)} \qquad \text{for } z > 0 \tag{7}$$

where

$$\log\overline{L}_\pm(\zeta) = \frac{-1}{2\pi i}\int_{-\infty\mp\epsilon}^{+\infty\mp\epsilon} dz'\,\frac{\log\overline{K}(\zeta')}{\zeta' - \zeta},$$

so that $\overline{L}_+(\zeta)$ and $\overline{L}_-(\zeta)$ are analytic in the upper and lower half of the ζ-plane, respectively, and $\overline{K}(\zeta) = \overline{L}_-(\zeta)/\overline{L}_+(\zeta)$. If C_0 is deformed so that it is wrapped around the right branch cut, and for kz not too small, the contribution to the integral of (7) comes mainly from near $\zeta = k$, so that

$$i(z) = \frac{k}{\pi i}[\overline{L}_+(+k)]^{-2}\int_{C_2}\frac{e^{i\zeta z}}{(k^2 - \zeta^2)\overline{K}(\zeta)}d\zeta. \tag{8}$$

Comparing (8) with (1) gives

$$i(z) = -RI_\infty(z) \tag{9}$$

for large kz, where

$$R = \frac{\zeta_0}{2\pi}[L_+(k)]^{-2}.$$

The approximate value of $[\overline{L}_+(k)]^2$ can be shown to be

$$[\overline{L}_+(k)]^2 = \frac{1}{2}\,\frac{1}{C_w + i\dfrac{\pi}{2}}. \tag{10}$$

What (9) means is that, when away from the end, the reflected current distribution behaves just like that of the current on an infinitely long antenna driven by a fictitious generator of voltage $-R$ located at the point of reflection.

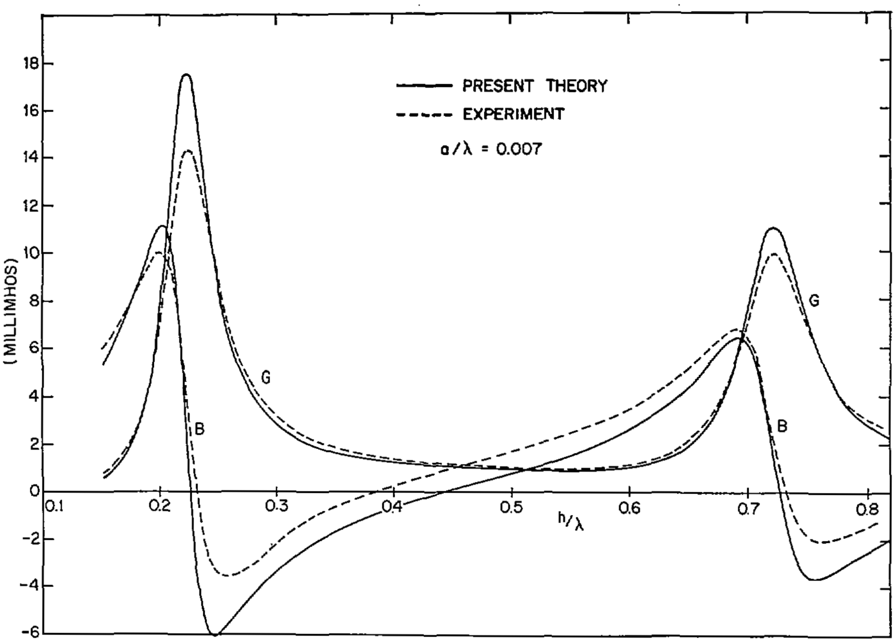


Fig. 3. Experimental and theoretical admittances.

IV. Current on a Finite Dipole Antenna

The foregoing analysis suggests that the current on a finite antenna may be expressed as the superposition of the outgoing wave, which is $I_\infty(z)$, and the reflected waves, which are proportional to $I_\infty(h_1+z)$ and $I_\infty(h_2-z)$, respectively. Let the coordinate be chosen so that the ends of the antenna are at $z=-h_1$ and $z=h_2$, and the driving point is at $z=0$; then

$$I(z) = I_\infty(z) + C_d I_\infty(h_1 + z) + C_u I_\infty(h_2 - z). \quad (11)$$

Assume the reflection coefficient $R(L)$ at the end of an antenna to be a slowly varying function of the parameter L (the distance between the generator and the reflecting end), so that the limiting values of R as L goes to infinity (as obtained in the previous section) can be used universally for all L. The validity of this assumption is discussed in detail in Shen *et al.* [19]. Now, since the amplitude of the incident wave at $z=-h_1$ is $I_\infty(h_1)+C_u I_\infty(h_1+h_2)$, while that of the reflected wave is C_d,

$$-R = \frac{C_d}{I_\infty(h_2) + C_u I_\infty(h_1 + h_2)}. \quad (12)$$

Similarly, at $z=h_2$,

$$-R = \frac{C_u}{I_\infty(h_2) + C_d I_\infty(h_1 + h_2)}. \quad (13)$$

Solving (12) and (13),

$$C_d = -R\left[\frac{I_\infty(h_1) - R I_\infty(h_2) I_\infty(h_1 + h_2)}{1 - R^2[I_\infty(h_1 + h_2)]^2}\right] \quad (14)$$

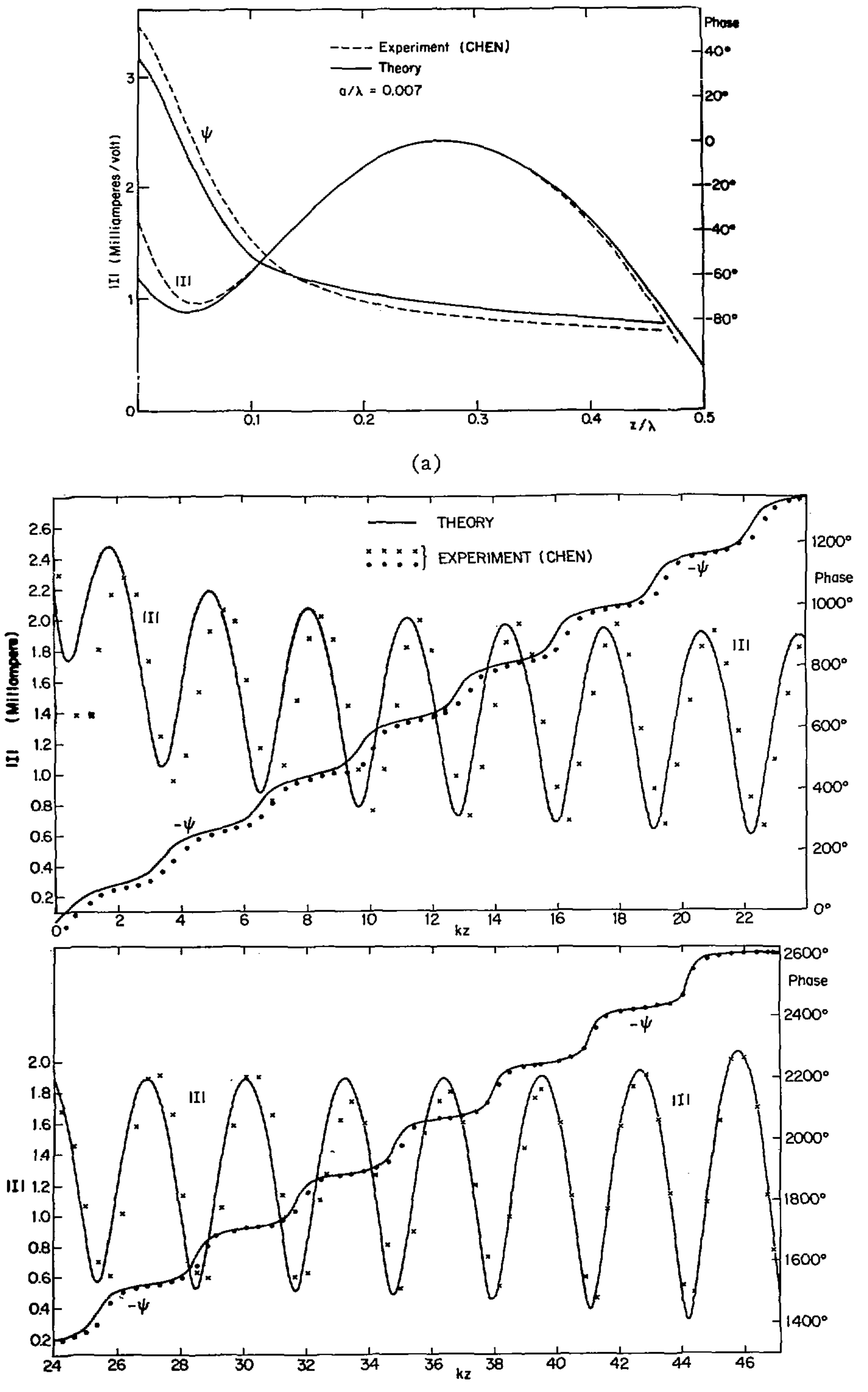
$$C_u = -R\left[\frac{I_\infty(h_2) - R I_\infty(h_1) I_\infty(h_1 + h_2)}{1 - R^2[I_\infty(h_1 + h_2)]^2}\right], \quad (15)$$

where R is given by (10) and $I_\infty(z)$ is given by (6).

The input admittance and the current distribution in a center-driven antenna calculated from (11) are compared with experimental data which are obtained from the works of Mack [20] and Chen [21], with antenna lengths varying from 0.5 to 15 wavelengths. Typical curves are shown in Figs. 3 and 4. The experimental data for current distribution obtained by Mack are normalized at the driving point, while those obtained by Chen are normalized at quarter-wavelength from the end of the antenna. Note that in Chen's setup the antenna was supported by a polyfoam slab. The agreement between present theory and experiment is seen to be remarkably good. Some of the discrepancy may be due to the experimental difficulty of absolute measurements and, in cases of long antennas [Fig. 4(b)], as pointed out by Chen, the presence of the polyfoam is responsible for a small increase in the effective length of the antenna.

The current distribution in a symmetrically loaded center-driven antenna can be obtained by superposition using (11). The theoretical result is compared with Altshuler's experiment on a resistively loaded traveling-wave antenna [22]. Comparisons were made for resistively loaded antennas with lengths varying from 1.0 to 5.5 wavelengths. The agreement between theory and measurement is found to be satisfactory. Some of the results are shown in Fig. 5.

In Fig. 3, it is observed that the admittance of a symmetrically driven antenna calculated by present theory agrees with experiment when h is greater than 0.15λ. If this length is accepted as a limit to the present theory, then in the asymmetrically driven case, (11) should be good when the generator is no less than 0.15λ away from either end of the antenna.



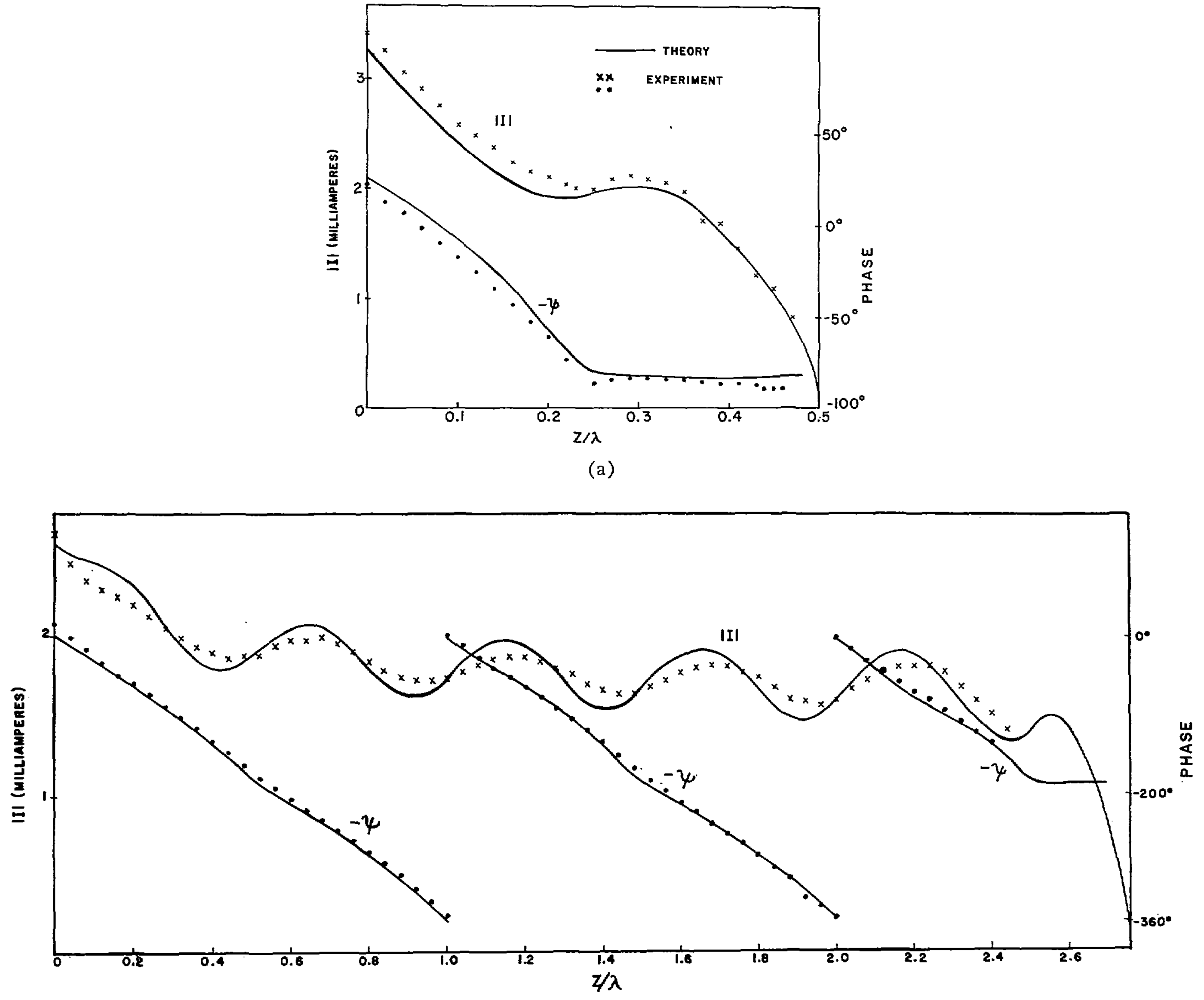
(a)

(b)

Fig. 4. Current distribution. (a) $h = 0.5\ \lambda$. (b) $h = 7.5\ \lambda$.

V. Conclusions

The foregoing discussion seems to support the following picture of a dipole antenna. An outgoing traveling wave of current is generated along the dipole antenna when it is driven by a time-harmonic source. It travels along the two arms of the dipole with a speed almost equal to the speed of light, and decays slowly in a manner described by (6), as a result of radiation. It is reflected at the ends of the dipole with the reflection coefficient given by (10). After it is reflected, the current wave travels in the opposite direction with the same speed and decays in the same manner as before. The current distribution on the antenna is just the result of the superposition of the outgoing current wave and all the reflected waves. This description of the current along a dipole antenna is analogous to that for a lossless transmission line, for which the traveling wave of current does not



(a)

(b)

Fig. 5. Current distribution. (a) $h=0.5\ \lambda$, $h_L=0.25\ \lambda$. (b) $h=2.75\ \lambda$, $h_L=2.5\ \lambda$.

decay. This qualitative picture is, of course, not new [5]; the main point of the present paper is that quantitatively accurate results are obtained when a good approximation of the current in an infinite antenna is used. Numerical results are quite accurate when $h \geqq 0.15\lambda$.

APPENDIX

In this appendix, an approximation to the following integral is demonstrated. This approximating method is used in deriving (3) from (2):

$$I = \int_0^\infty \frac{e^{-\eta z}}{\eta} f(\eta)\,d\eta, \tag{16}$$

where $f(\eta)$ is slowly varying and is zero at $\eta=0$ so that the integral is well defined. I is to be approximated by the integral

$$I \cong \int_0^{D'/z} \frac{d\eta}{\eta} f(\eta). \tag{17}$$

The task is to find D'.

This problem is approximately equivalent to determining the constant D in the following equation:

$$\int_{D/z}^\infty \frac{d\eta}{\eta} e^{-\eta z} = \int_0^{D/z} \frac{d\eta}{\eta} (1 - e^{-\eta z}). \tag{18}$$

Equation (18) is equivalent to

$$\int_{D}^\infty \frac{d\eta}{\eta} e^{-\eta} = \int_0^{D} \frac{d\eta}{\eta} (1 - e^{-\eta}). \tag{19}$$

Since

$$-\gamma = \int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} \tag{20}$$

(see Erdelyi[1]), substituting (19) in (20) yields

$$-\gamma = \log D$$

or

$$D = e^{-\gamma}.$$

This is an exact solution. If $f(\eta)$ is slowly varying, it should be a good approximation to let $D' = D = e^{-\gamma}$.

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